Retirement Income Analysis
with scenario matrices

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7. The Market Portfolio

Riskless and Risky Assets

As indicated earlier, we will focus much of our analysis of investment alternatives on two key assets. The first, providing riskless real returns, was covered in Chapter 6. The second is a portfolio of securities that provides uncertain future real returns. But not just any such portfolio. Rather, we use a practical approximation of a theoretical construct termed *the market portfolio.*

In a simple world, the market portfolio would include every publicly traded security, with each held in proportion to the total amount outstanding. An investor could hold his or her version of the market portfolio by purchasing x% of the outstanding shares of every traded stock and x% of the outstanding number of bonds for every traded bond, where x is the ratio of his or her invested wealth to the total value of the amounts invested by everyone.

Importantly, it would be possible for each investor to hold such a market portfolio. The market would *clear,* since for each stock or bond the total quantity demanded would equal the amount available. Moreover, a recommendation that each investor put his or her "at risk" assets in the market portfolio would be *macro-consistent* advice, in the sense that everyone could implement such a strategy.

“Smart” Investment Strategies

Many investment firms and advisors argue that some portfolio composition other than that of the market is “smarter” and will provide better outcomes for any investor (although different firms and advisors tend to differ in their choices of superior investments). One hears of strategies with names such as “smart beta”, “factor tilt”, “momentum”, “value” “small capitalization” and on and on.

Let's say that the amount you wish to invest in a risky portfolio is x% of the total value of all the securities in the market and that your investment advisor advocates that you *overweight* (hold more than x% of) certain “underpriced” securities , *market-weight* (hold x% of) those that are “correctly priced”, and *underweight* (hold less than x% of) those that are “overpriced”. This, he or she says, is a portfolio with better risk and return characteristics than the market portfolio.

Perhaps it is, and by holding it you will indeed be smart. But if so, then those holding the market portfolio must be dumb. And those who underweight the securities that you overweight and overweight the securities that you underweight are even dumber. If this is the case, one might assume that sooner or later both groups will recognize their mistakes and try to buy the underpriced securities and sell the overpriced ones. But of course every buyer needs a seller and every seller needs a buyer. The net result will be for the prices of the formerly underpriced securities to increase and the prices of the formerly overpriced securities to decrease until every security is “correctly priced”. At this point it will be smart to hold the market portfolio. In this sense a strategy that can successfully “beat the market” will carry the seeds of its own destruction.

Data Mining

Someone once said that if you torture a body of data long enough, it will eventually confess to something. This is especially true of financial data. We now have massive amounts of information on the characteristics and returns of hundreds of thousands of securities over many years as well as computers and programs able to determine the returns over time from myriad investment approaches. Not surprisingly, by testing thousands if not millions of portfolio management systems it is possible to find one or more that would have produced superior, or even spectacular results. Many investment analysts, having discovered such systems, find it impossible to resist the temptation to create a mutual fund, exchange-traded fund or investment advisory practice using one of them, with documents showing "backtests" indicating its superiority in the past.

One skeptic of such approaches famously said that he had never met a backtest in such a document that he didn't like. Another said that if someone brought an investment product to market with a backtest showing that it would have performed poorly in the past he might invest in it, just to reward candor.

Financial history is replete with examples of cases in which an investment approach with superior past performance fails to "beat the market" in the future. In some cases, this may have been due to a focus on "noise" in a body of historic data. In others, is may be a change in market prices caused by the realization by sophisticated investors that some security prices were inappropriate in the past, resulting in corrections leading to more appropriate valuations.

In any event, it pays to be very skeptical indeed of schemes that purport to be able to "beat the market".

The Arithmetic of Active Management

Imagine a scenario in which you have in one auditorium professional investment managers of funds that hold all the stocks of country *Z,* which uses the dollar for currency. On one side you have those who run *index funds*, each of which holds shares in each of the stocks in market proportions. On the other side you have those who run actively-managed funds (*active funds*), so named because their managers are actively investigating companies and industries in order to discern “underpriced” and “overpriced” stocks, then investing their funds accordingly. To keep the story simple, assume that individuals invest only through the funds managed by those in the room (although any individual managing his or her own investments could be included in the room without changing the key conclusions of this argument).

Now, assume that in the year just ended, the overall dollar return on the (value-weighted) market portfolio of all the stocks in country Z was 10.0%.

What was the return *before costs* for the fund run by index fund manager 1? Answer: 10.0%. The before-cost return for the fund run by index fund manager 2? Again, 10.0%. And so on. And what was the return before costs on each dollar invested with the index fund managers in the room? Clearly, 10.0%. And the return before costs on the sum of all the dollars invested in index funds? Also 10.0%.

Now, consider the active managers. Perhaps manager *A1* achieved a before-cost return of 15.0%. And manager *A2* had a before-cost return of 2.0%. Unfortunate manager *A3* had a really bad year, with a before-cost return of -8.0%. And so on.

But here is a crucial question. What was the before-cost return on the sum of all the dollars invested in the active funds? The answer is not difficult to determine. Before costs, if the return on the sum of the dollars invested in the market was 10.0% and the return on the sum of the dollars invested in the indexed portion of the market was 10.0%, then the return on the sum of the dollars invested actively must have been 10.0%. This is simple arithmetic.

Put another way:

> **Before costs, the return on the average *actively managed* dollar must equal the return on the average indexed (*passively-managed*) dollar.**

This is not derived from some complex equilibrium theory based on a host of unrealistic assumptions, just the rules of elementary-school arithmetic.

There is more. Investment management costs money and investors should be concerned with after-cost returns. So let's consider the impact of investment managed fees.

Index managers need to find the financial statements of companies in their market, the current prices of the securities they cover and the number of shares of bonds outstanding. Then they need to do some arithmetic operations, and buy or sell securities as needed when investors provide new funds or wish to cash out. Of course records must be kept, tax information provided, etc.. But for a large index mutual fund, the total cost per dollar invested can be very low. For example, the largest U.S. equity fund in mid-2015 was the Vanguard Total Stock Market Fund. For investments of more than $10,000, the annual fee paid by investors was 0.05% (5 cents per year per $100 invested).

Active managers do much more (that is why they are called active!). They study earnings reports, analyze industry positions, research new products and competitive firms, torture large bodies of historic data, visit industry executives, take people with useful information to sports events, etc. etc.). Moreover, they command larger salaries and bonuses than the clerks and possibly reclusive managers at passive funds. All this activity costs money. According to Morningstar, an firm that analyzes the fund industry, the average fee charged by U.S. large-capitalization actively managed funds in 2015 was 1.04% ($1.04 per $100 invested) .

This leads to one of the most important conclusions in investments:

> **After costs, the return on the average *actively managed* dollar must be less than the return on the average indexed (*passively-managed*) dollar.**

Clearly, a result that active investment managers hate to have publicized. But since I first made the point in a short article published in the Chartered Financial Analysts' own publication, *The Financial Analysts Journal (*January/February 1990), many empirical studies have provided results consistent with the assertion.

Of course, in a given period some active managers can beat an appropriate index strategy, even after costs. But it is difficult to do so over extended periods of time or with any consistency from year to year. And after costs, the difference between active and passive management can be large: for U.S. Stocks, perhaps as much as 1.00% per year.

The Arithmetic of Investment Expenses

Some investors consider the difference between an annual investment expense of 1.04% and 0.05% a small price to pay for purportedly "superior" investment choices. After all, they say, even though we might have 1% less to spend, we could do much better than average. But this is an incorrect calculation. Why? Because the extra cost is 1% *per year*.

In a subsequent paper ("The Arithmetic of Investment Expenses," *Financial Analysts Journal, March/April 2013*), I showed the impact of such an annual difference on the retirement savings of two individuals – one who had invested in a low-cost stock index fund, the other who had chosen a typical actively-managed stock fund, taking into account only the difference in investment management fees. The results depended, of course, on many factors, which were taken into account with simulations and sensitivity analyses. But the bottom line was that "... the odds are even that the frugal (index fund) investor will have over 20% more money to spend during retirement" than the non-frugal investor who chooses an average actively managed fund.

In a later paper, "The Arithmetic of 'All-in' Investment Expenses" (*Financial Analysts Journal*, January/February 2014), John Bogle (the founder of Vanguard) attempted to measure additional costs associated with active management, including a "conservative estimate" that the average active fund incurs a cost of 0.50% per year for transactions associated with security purchases and sales generated by estimates of changing mis-valuations.

As we will see it is a simple matter to reflect active management costs when generating a retirement income scenario matrix by simply entering a lower expected return than that of a market index fund; one can also increase the predicted risk. But the likely consequence can be anticipated. If, for example, the expected real return for the market is estimated to be 5%, a market index fund might be expected to return 4.9% or more after costs while an average actively managed fund with similar risk might be expected to earn 3.5%: 4.9% - (1.0% additional fee plus 0.5% added transaction costs). The active fund would thus return 3.5/4.9 or 71.4% as much real return each year, over a period of many years. With luck, an active management policy might make up this difference or even beat an index alternative after costs. But the odds are that a retiree with actively managed investments will have to accept a considerably lower standard of living than his or her neighbor who has chosen an index fund of comparable risk.

The Capital Asset Pricing Model

As we have seen, based solely on arithmetic, there are compelling arguments for investing in a low-cost index fund that tracks a widely diversified portfolio with holdings in market-value proportions. We turn now to the first of two theoretical arguments for choosing the most diversified such portfolio available: *the market portfolio*. Each argument is based on a highly simplified model of a capital market, and each abstracts from many aspects of the real world. As with any theory that abstracts from reality to focus on what are assumed to be the key aspects of a problem, one must judge the conclusions on their merits, not on the realism of the assumptions made in the model that produced them.

This section provides key aspects of the first approach, based on my 1959 PhD dissertation at UCLA, published five years later in the Journal of the American Finance Association as "Capital Asset Prices – A Theory of Market Equilibrium Under Conditions of Risk", in *The Journal of Finance*, September 1964. It is now included in most academic investment textbooks, often as the only detailed theory of equilibrium in capital markets, then usually followed by a discussion of many reasons why it may not fully describe real security markets. Shortly after its introduction, others began to describe it as the *Capital Asset Pricing Model*, or *CAPM*, and the name stuck.

A key assumption of the CAPM is that investors think about the possible future return on a security or portfolio as a probability distribution and that they are concerned only with the *mean* and *standard deviation* of such a distribution. The mean, or *expected return* is computed by weighting each possible return by its probability, then summing. The standard deviation is computed by first finding the deviation of each possible return from the mean, squaring it, weighting it by its probability, then summing to find the *variance*. The square root of the variance is the standard deviation.

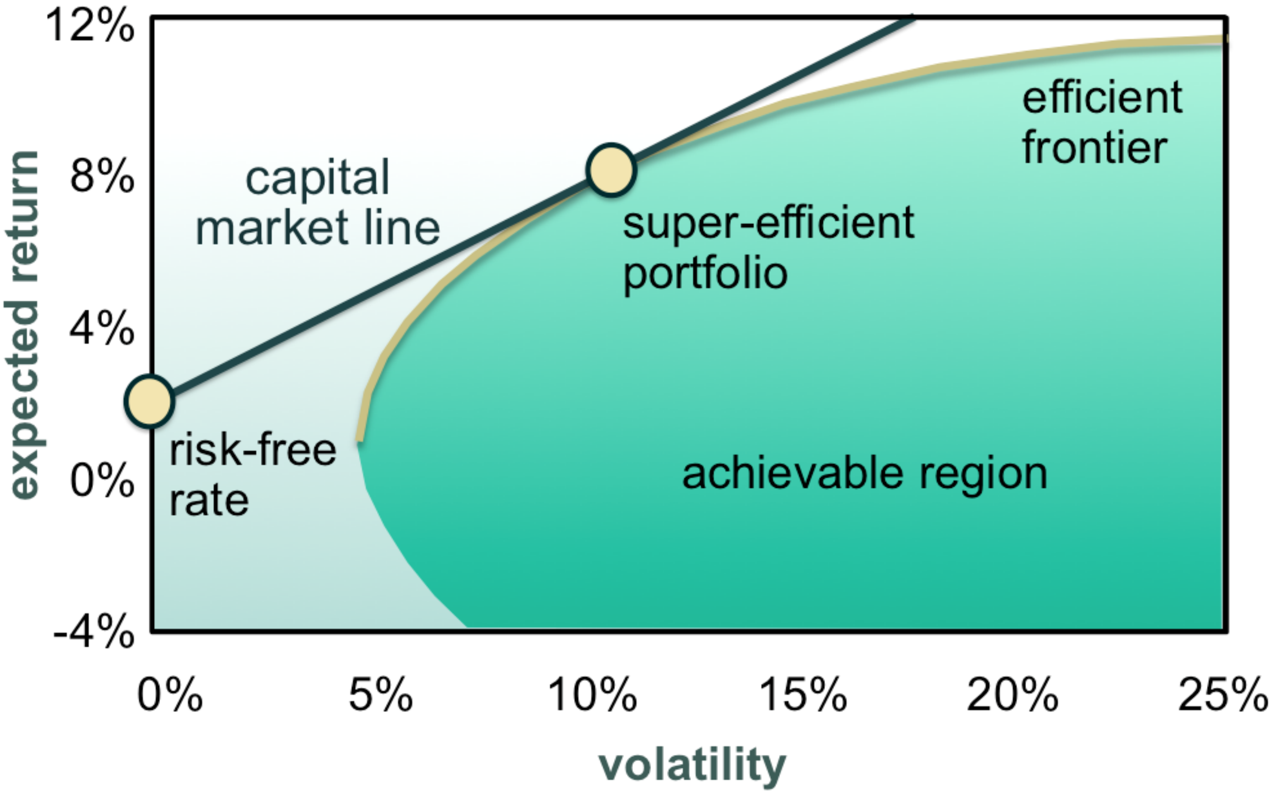
This *mean-variance* approach as a tool for portfolio construction was initially advocated by Harry Markowitz in his famous works on the choice of optimal portfolios, given a set of estimates of security mean returns, standard deviations and correlations between pairs of securities – first in "Portfolio Selection", J*ournal of Finance*, March 1952 and then in his 1959 book "*Portfolio Selection: Efficient Diversification of Investments."*

The CAPM assumes that investors concentrate on investment return means and variances and use the portfolio optimization methods that Markowitz advocated. Formally, it adds the very strong assumption that everyone agrees on the means, variance and correlations for individual securities. Given this, the model determines the conditions for a set of security prices that would make investors collectively wish to hold the available securities, since only for such prices would the quantity of each security demanded equal the quantity supplied, and the overall capital market would be in *equilibrium*.

Using standard terminology, Markowitz' portfolio theory is normative (prescriptive) --"what you should do", and the CAPM is positive (descriptive) – "what is".

For reasons to be given later, we will not rely on mean-variance analysis or the CAPM, and thus will only summarize here its key result concerning the market portfolio.

An online search for *capital market line images* will produce a great many diagrams. The one below, from *RiskEncyclopedia.com*, is one of the more colorful.

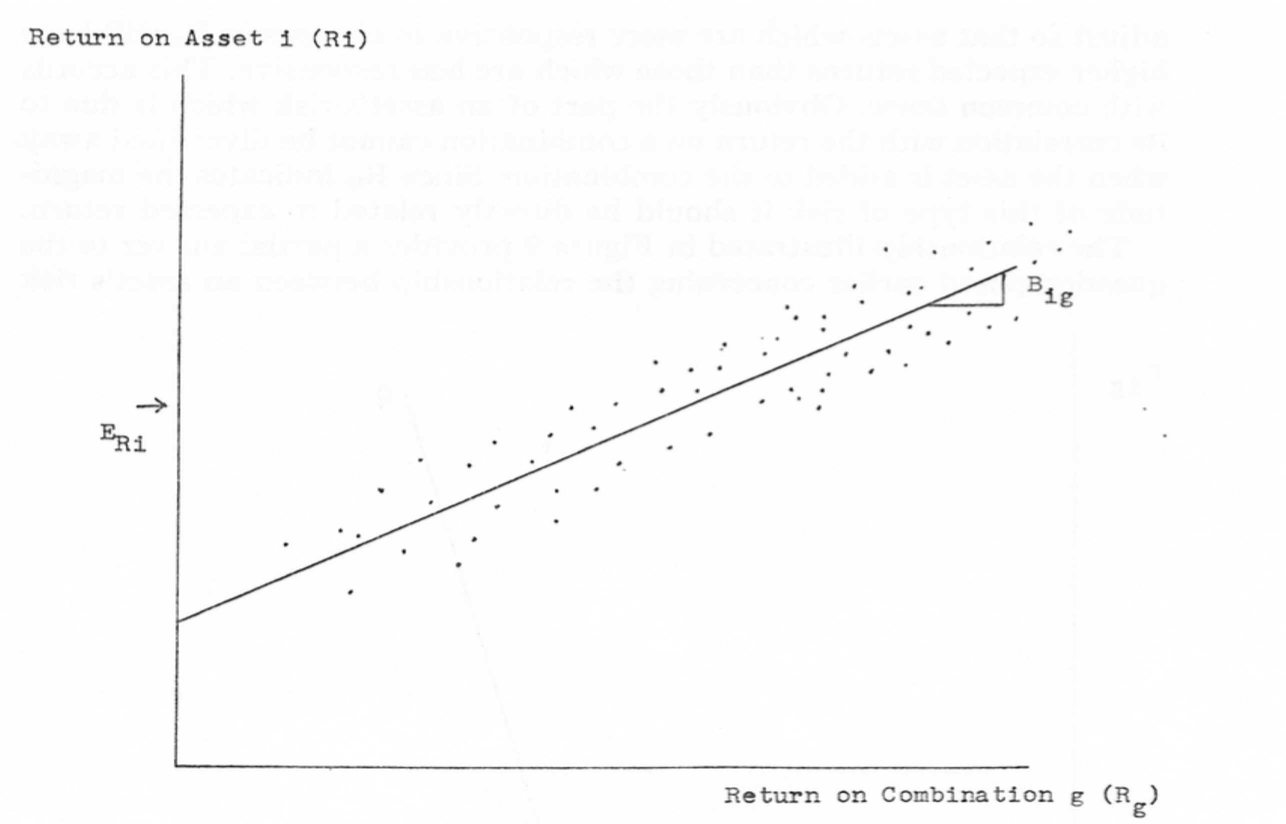


In this diagram, the vertical axis plots expected (mean) return and the horizontal axis the standard deviation of return (here, called "volatility"). The darker area (here, the "achievable region") represents a region within which every possible portfolio of risky securities would plot, each as one point. The curved line at the top of the region is the *efficient frontier*, developed by Markowitz. Each portfolio on this frontier provides the greatest possible expected return for a given level of risk, if (and only if) only portfolios of risky securities are considered.

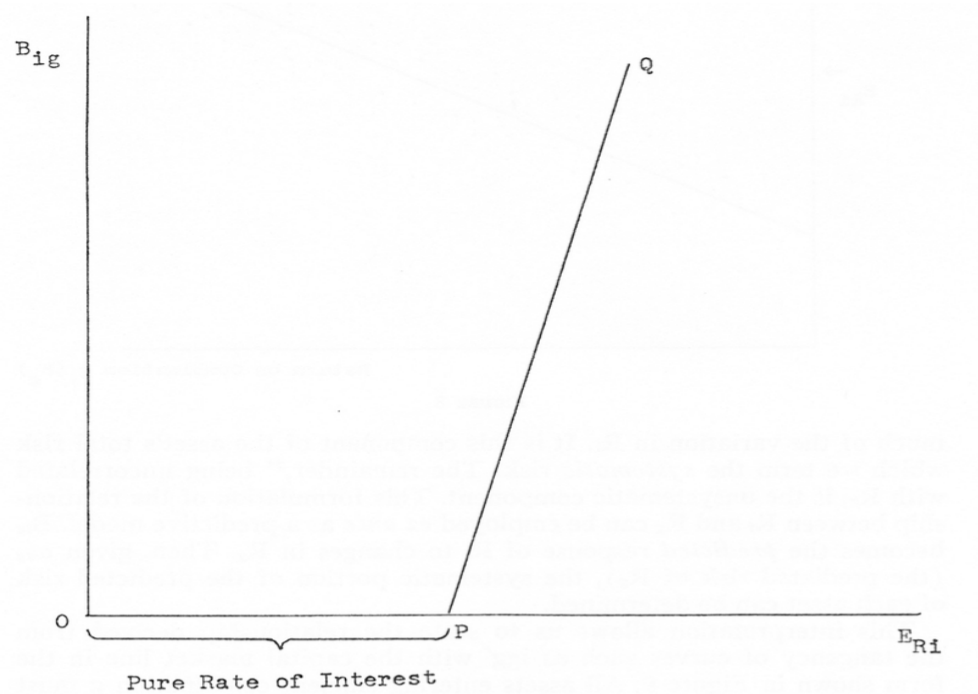
But what if one can invest in a risk-free security? It will plot on the vertical axis at a point representing the *risk-free rate*. As shown by James Tobin in "Liquidity Preference as Behavior Towards Risk" (*Review of Economic Studies*, Feb. 1958), simple algebra shows that by combining such a risk-free asset with any risky portfolio, one can attain a point on a line connecting their two locations. Moreover, if one could borrow funds at the risk-free rate, it would be possible to attain a point on the extension of the line to higher risks and expected returns. In such a world, there would be only one desirable portfolio of risky securities – the one that falls at the point where a line from the risk-free rate is tangent to the efficient frontier (here, called the "super-efficient portfolio"). And, as I argued in my 1964 paper, for there to be equilibrium, in equilibrium this would have to be the market portfolio of all risky securities, held in market proportions.

I called the line drawn from the risk-free asset point to the point for the market portfolio and beyond the *capital market line*, since in this setting it shows the maximum expected return obtainable for any given level of risk. The slope of the line is the ratio of (1) the expected return on the market portfolio minus the risk-free rate divided by (2) the market portfolio's risk (standard deviation). In a paper also based on my dissertation ("Mutual Fund Performance", in *The Journal of Business of the University of Chicago*, January 1966), I called this the *reward-to-variability ratio,* but others chose to refer to it as the *Sharpe Ratio*, which it remains to this day. While simple to the point of being simplistic, it takes into account both risk and return. In an *ex ante* setting such as the CAPM, the numerator is the expected return over and above the risk-free rate, while the denominator is the standard deviation, a measure of risk. In the CAPM, the market portfolio will have the greatest possible Sharpe Ratio and hence be the best investment. Many use Sharpe Ratios for *ex post* analyses (as did I in the original article), dividing average returns over a riskless rate by realized standard deviations. As will be seen, we will need to consider historic data, but only to estimate the future Sharpe Ratio of the market portfolio.

The CAPM had another result, summarized in two figures taken from the 1964 article. The first illustrates the calculation of an asset's *beta value,* defined as the covariance of its returns with those of an efficient portfolio, (here, *g*) divided by the variance of that portfolio. As the diagram shows, this can be thought of as the slope of a regression line with the two returns on the axes.



The second graph provides the key equilibrium result – the expected return on any asset (i) will be a function solely of its beta value, the function will be linear, and it will go through the points representing the risk-free asset (here, the "pure rate of interest" *P*) and an efficient portfolio (here, *g*). I called this relationship the *security market line*. Those familiar with the literature may find this unusual, since it is now conventional to put expected returns on the vertical axis, and beta values on the horizontal axis.



The CAPM results depend on a host of very strong assumptions. Moreover, its predictions are about future expectations and covariances of return distributions, not realized average returns and beta values. Even if the theory were true *ex ante*, the *ex post* empirical relationship could differ.

For valuing investment strategies we will rely on a more general approach that will developed at length in the next chapter. Like the CAPM, it assumes that the market portfolio is an efficient vehicle for holding risky assets and that only market-related risk will be rewarded. But, as we will see, the nature of the relationship between reward and risk will be somewhat different.

We now turn to practical issues associated with investing in a market portfolio.

Components of the Market Portfolio

In the early days of financial market equilibrium theories, limited data availability led most researchers to use indices of the returns on stocks as measures of the performance of of "the market". But this was less than ideal. Consider publicly held corporate securities. A company may finance its activities by issuing bonds and stocks. The particular combination utilized should not affect the overall claims of the public on the corporation's earnings. Certainly the market portfolio should include both corporate stocks and bonds held by the public.

But what about government bonds? Some would say that the net public interest in such securities is zero, since a bond held by an investor is his or her asset and the obligation to pay interest and principal is a liability for those who will have to pay taxes in the future to cover such payments. An alternative view is that a government bond represents a claim on the future earnings of taxpayers and hence an investment in their human capital. While each argument has merits, we take the latter position.

There are of course other types of investments. Some, such as traded options, have net values of zero, with one public investor on one side of an agreement and another on the other. Others, such as stocks of private corporations, are not liquid and thus not particularly suitable for a portfolio that may need to be liquidated on relatively short notice.

For better or worse, it seems sensible to consider a portfolio of publicly traded bonds and stocks as a surrogate for the market portfolio of investment theory. At the risk of seeming parochial, we will concentrate on low-cost investment products designed for investors in the United States (and possibly others who choose to purchase U.S. Dollar-based investment funds). Since it would be daunting and undoubtedly too expensive for retirees to purchase thousands of securities directly, we will focus on low-cost mutual funds or exchange-traded funds with appropriate holdings.

To simplify the task, consider the four following key categories:

	U.S.	**Non-U.S.**
Bonds		
Stocks		

Our goal is to select investment vehicles and appropriate amounts to be invested in each sector to obtain a good approximation for the *World Bond/Stock Portfolio, (*hence: *WBS)*.

Low-cost Index Funds

A search for low-cost funds that can well represent one or more of our four categories leads to a few providers. Based on its size and low costs, we will choose the Vanguard Group – the only investment fund company organized as a non-profit entity. This, plus the large amounts invested in its funds, allow for very low fees. And, while both Vanguard and others offer exchange-traded funds tracking some of our categories, as this is written in 2015, none has lower fees than the comparable Vanguard mutual fund. More generally, mutual funds offer record-keeping, tax filing services and other advantages for the long-term investor. For all these reasons, we will use Vanguard mutual funds.

One might assume that Vanguard and/or other companies would offer a single fund tracking the returns on a world bond/stock portfolio. Alas, this is not the case. Vanguard has a fund that covers both U.S. and Non-U.S. stocks, but the fees are higher than a combination of their U.S. stock fund and Non-U.S. stock fund. Instead, we follow the advice given in mid-2015 on the Vanguard web site:

> *In fact, the right balance of four of our broadest index funds could give you a complete portfolio, with full exposure to U.S. and international stock and bond markets.*
>
> *Get details on:*
>
> *Vanguard Total Bond Market Index Fund*
> *Vanguard Total International Bond Index Fund*
> *Vanguard Total Stock Market Index Fund*
> *Vanguard Total International Stock Index Fund*

We do so, despite the parochialism reflected in the names of the first and third fund, which hold only U.S. bonds and stocks, respectively.

Each fund is available via either *Investor* or *Admiral* shares. The latter require an initial investment of $10,000 or more, but fees are lower. The table below provides the ticker symbols and expense ratios in 2015 for the Admiral shares.

	U.S.	**Non-U.S.**
Bonds	VBTLX: 0.06% per year	VTABX: 0.14% per year
Stocks	VTSAX: 0.05% per year	VFWAX: 0.13% per year

Expense ratios are determined in part by the value of the total assets invested in a fund. Not surprisingly, VTSAX, the world's largest mutual fund at the time, had the lowest costs, but VBTLX was only slightly more expensive. The higher fees for Non-U.S. Funds reflect both their smaller sizes and the costs associated with investing outside the country (plus, for VTABX additional financial transactions described below). While one might choose to underweight the more expensive funds to take into account their higher costs, this would require assumptions about the joint probability distribution of future returns – a difficult task that we will not undertake.

Each of our chosen funds is an *index fund,* intended to track an independently-provided index of returns for securities in its domain. Vanguard provides two key statistics on the extent to which the historic returns of each fund reflected those of its underlying index. *Beta* is the slope of a regression line with the index return on the horizontal axis and the fund return on the vertical axis, *R-squared* is a measure of the extent to which the points in such a diagram fall on the line. Ideally, a fund should have an historic beta of 1.0 with an associated R-squared of 1.00.

The fund benchmarks and their historic statistics (rounded to two decimal places) at the end of April, 2015 are shown below.

VBTLX: Vanguard Total Bond Market Index Fund Admiral Shares
Index: Barclays U.S. Aggregate Float Adjusted Index
Beta: 1.02
R-squared: 1.00

VTABX: Vanguard Total International Bond Index Admiral Shares
Index: Barclays Global Aggregate ex-USD Float Adjusted Regulated Investment Company Index Currency Hedged
Beta: N/A (insufficient history)
R-squared: N/A (insufficient history)

VTSAX: Vanguard Total Stock Market Index Admiral Shares
Index: The Center for Research in Security Prices U.S. Total Stock Market Index
Beta: 1.00
R-squared: 1.00

VFWAX: Vanguard FTSE All-World ex-U.S. Admiral Shares
Index: FTSE All-World ex U.S.
Beta: 1.01
R-squared: 0.99

Since VTABX was first funded in May 2013, Vanguard did not show statistics less than two years later. But the records indicate that each of the other three funds tracked the returns on its index remarkably well; moreover, the cumulative return of VTABX since inception was close to that of its index.

But does each of the underlying indices represent the returns on the average dollar (or other currency) invested by the public in its domain? Index providers increasingly have tried to represent the returns on publicly available securities by focusing on "free float", that is, omitting from their calculations holdings not "freely available" for purchase by outside investors. Here are representative descriptions from the three index providers.

> *Barclays US Aggregate Float-adjusted index ".. adjusts for US Federal Reserve holdings of US MBS pass-throughs and US agency bonds, in addition to Treasuries." Barclays Float-adjusted Pan-European Aggregate ".. adjusts the amount outstanding of Gilts for Bank of England purchases" and their Float-adjustedAsia-Pacific Aggregate Index "... adjusts for JGB par amount outstanding for Bank of Japan purchases."*

> *The CRSP U.S. Equity indexes are ".. free float capitalization-based indexes. Float shares outstanding represent the total shares outstanding less any restricted share, which are defined as those held by insiders or stagnant shareholders – including, but not limited to: board members, directors and executives' government holdings, employee share plans and corporations not actively managing money."*

> *The FTSE equity indices reduce a company's weight in an index ".. to take account of restricted holdings of the company's shares that are not freely available for purchase by outside investors. Examples of such restricted holdings include strategic investments by governments and other companies, directors, founder family holdings, holdings of other major investors who have influence over the direction of the company, and shares held by investors with restrictions on trading ('lock-ins')".*

To reduce turnover, most providers of float-adjusted indices change float adjustments episodically, using bands within which no changes are made; they also employ various types of rounding. Overall, it seems consistent with the spirit of "the market portfolio" to use both the market values and returns on such float-adjusted indexes and to invest in the funds we have chosen to track them.

Not all securities are included in three of the indices. The Barclays indices include only bonds rated BBB or higher. The FTSE All-World ex-U.S. Index excludes small-capitalization stocks but nonetheless covers 90% to 95% of the available market, according to the FTSE fact sheet (although it is likely that at some time, Vanguard will replace this with the corresponding FTSE Global index, which includes small-capitalization stocks).

One aspect of the Barclays Non-U.S. bond index, and of the Vanguard fund that tracks it, merits mention. Both are “currency hedged”. In effect, VTABX provides investment in non-U.S. Bonds plus monthly side-bets on changes in exchange rates between the U.S. Dollar and the home currencies of the bonds in the portfolio. This is accomplished by committing to exchange foreign currencies and dollars at a pre-specified exchange rate a month hence. There will of course be traders or investors on the other side of each of these transactions (undoubtedly including some Europeans, Japanese, and British), who may well consume goods produced in other countries. To be sure, if exchange rates always reflected the relative costs of buying goods and services with different currencies so that *purchasing power parity* held, real returns on unhedged positions might be little affected by changes in such rates. But at best, this is likely to be true only in the long run.

One might ask why non-U.S. Bond exposures should be hedged but not non-U.S. Stock exposures. A partial justification would be the much greater uncertainty about the value of stock positions a month hence, making currency hedges based on the estimated value a month hence of the securities currently held far less than perfect. Happily, despite providing this additional feature, the expense ratio for VTABX is considerably lower than the costs for Non-U.S. bond funds that do not employ currency hedging.

Market Capitalizations

We now have funds with which to build a world bond/stock portfolio. But how much should be invested in each? Presumably, the proportions should reflect the relative values of the publicly-held securities in the underlying domains. Of course, to weight security returns appropriately, index providers have to compute the market capitalization of each one and divide by the sum – the value that we seek. Almost all index purveyors episodically provide the total values of such market capitalizations to index fund companies, financial data providers and paid subscribers to their data services. But not all, publish the market capitalizations of their indices at the end of each month or quarter online for public access. CRSP publishes a free online Factsheet showing the market capitalization for its U.S. stock index at the end of each quarter. FTSE provides free online Factsheets for its Non-U.S. Stock indices at the end of each month. Unfortunately, Barclays' online factsheets do not include information on the market capitalizations of its indices. However, Citibank does so for a set of somewhat similar bond indices.

The Citibank US Broad Investment-Grade Bonds Index (USBIG) includes "... US Treasury, government sponsored, collateralized, and corporate debt providing a reliable representation of the US investment-grade bond market ... with bonds rated from AAA to BBB inclusive by Standard and Poor's Financial Services." The Citibank World Broad Investment-Grade Bond Index (WorldBIG) includes "... government, government-sponsored/supranational, collateralized, and corporate debt" and excludes lower-rated bonds. We will use the difference between the values of these indices as a measure of the value of Non-U.S. Bonds.

Unfortunately the Citi bond indices are not float-adjusted. However, they appear to cover fewer bonds than those included in the Barclays indices, so their market capitalizations could be reasonably close to those of the latter. In any event, they are available so we will use them.

To summarize, these are the indices we will employ:

	U.S.	**Non-U.S.**
Bonds	Citi: USBIG	Citi: WorldBIG - USBIG
Stocks	CRSP U.S.	FTSE All-World ex U.S.

Here are the values of the four components and their sums as of June 30, 2015:

	U.S. ($ Trillion)	Non-U.S. ($ Trillion)	Total ($ Trillion)
Bonds	17.04	16.83	**33.87**
Stocks	22.59	18.88	**41.47**
Bonds + Stocks	**39.63**	**35.71**	**75.34**

Dividing by the total value, gives the proportions for the WBS fund at the time:

	U.S. (%)	Non-U.S. (%)	Total (%)
Bonds	22.62	22.34	**44.96**
Stocks	29.98	25.06	**55.04**
Bonds + Stocks	**52.6**	**47.4**	**100.00**

It is striking that the U.S. stock portfolio, often used as a surrogate for the market portfolio was, in this broader view, only roughly 30% of the total value of all the world stocks and bonds. Note also that at the time, the value of stocks exceeded that of bonds, falling almost midway between the often recommended 60/40 mix of stocks and bonds and the somewhat less common advice to hold 50/50 proportions.

These are, of course, only somewhat rough estimates of values at a point in time. Moreover, while the FTSE and Citi index capitalizations are published monthly, the CRSP values are only available as of the end of each quarter. Moreover, some of the values are not available until the second week after the end of a quarter. Implementation thus requires additional measures to compute the amounts of each fund to be held both initially and subsequently, subjects to which we turn next.

Initiating and Rebalancing the Portfolio

To begin, assume that you are initiating an investment in the WBS portfolio. The first step is to determine the proportions that applied at the end of the prior quarter, using the fact sheets described in the previous section. Next you would need to determine the "total return" for each fund from that date to the present. This is easily done with data from the *Yahoo Finance* site : use the ticker symbol and select *historical prices,* then find the "Adj(usted) Close" prices for the end of the prior quarter and the most recent date available. These will be the same as the unadjusted prices for the latest date but may differ for earlier dates, to allow for dividends and distributions. The ratio of the two prices will be the ratio of values of the holdings at the two dates for an investor who chooses to reinvest a fund's dividends and distributions in the same fund.

The second step is to multiply each of the proportions at the end of the prior quarter by the ratio of adjusted closing prices for the two dates, then divide each of the four values by their sum. This will provide the appropriate percentages to be invested in the funds. Multiply each by the total to be invested in the portfolio, then place orders to invest each of these dollar amounts in the fund in question. By convention, an order to purchase mutual fund shares will be executed at the closing net asset value per share determined after the order is placed, so you will not achieve the precise proportions desired. But differences are likely to be small and in any event cannot be avoided.

The following MATLAB program did the computations for trades placed on July 13, 2015.

```
% wbsInitialize.m
% computes trades to be made to initialize a WBS portfolio
% market capitalizations: June 30, 2015
% last historic prices: July 10, 2015

% format for matrices:
%    U.S. Bonds    Non-U.S. Bonds
%    U.S. Stocks   Non-U.S. Stocks

% INPUTS
  % market capitalizations at end of prior quarter ($ trillions)
    EOQmarketCaps = [ 17.04  16.83;  22.59  18.80 ];
  % Historic adjusted prices, end of prior quarter
    EOQprices =          [ 10.72  20.90;  52.10  30.20 ];
  % Historic adjusted prices, last price date
    currentPrices =      [ 10.69  20.88;  52.40  30.08 ];
  % total amount in dollars to be invested in WBS
    amountInvested = 100000;

% COMPUTATIONS
  % compute End Of Quarter market proportions
    EOQproportions = EOQmarketCaps / sum(sum(EOQmarketCaps));
  % compute current/EOQ price ratios
    priceRatios = currentPrices ./ EOQprices;
  % compute revised market proportions
    products = EOQproportions .* priceRatios;
    revisedProportions = products / sum(sum(products));
  % compute desired market values
    desiredValues = revisedProportions * amountInvested;
  % compute amonts to trade
    trades = round( desiredValues );

% SHOW RESULTS
   disp(' Dollar amounts to be traded' );
   disp( trades );
```

In this case the results were:

Dollar amounts to be traded	
22581	**22344**
30192	**24884**

Since July 11, 2015 was a Saturday, the trades would be executed at net asset values determined after the markets closed on Monday July 13th.

Once you have initialed a WBS portfolio, you will need to consider periodically rebalancing the proportions invested in the funds. Many, if not most, financial advisors recommend that a mix of asset classes be maintained by periodically revising holdings to a fixed set of proportionate holdings or to proportions dictated by a "glide path" with a predetermined asset mix for each future period. But, as I argued in "Adaptive Asset Allocation Policies," *Financial Analysts Journal*, May/June 2010, any such policy can only be justified by assuming that capital markets are inefficient a particular way.

To see this, consider a simple policy calling for a mix of 60% stocks and 40% bonds. Now assume that stocks outperform bonds so the portfolio proportions change to 65% stocks and 35% bonds. To rebalance to a 60/40 mix, one would have to sell some stock holdings and buy bonds with the proceeds. But not everyone can sell stocks and buy bonds, since for every buyer there must be a seller. More generally, rebalancing to predetermined mixes will require selling *relative winners* and buying *relative losers,* and for this to be possible, some other investor or investors must be buying relative winners and selling relative losers. If the former strategy is smart, the latter must be dumb. And the intelligence of investors who "buy and hold" must fall somewhere in the middle.

In investment jargon, selling relative winners and buying relative losers is termed a "reversal" policy, while that of selling relative losers and buying relative winners is called a "momentum strategy". But neither is macro-consistent. Despite protestations to the contrary, financial advisors who recommend rebalancing periodically to pre-determined value proportions are active managers who should recognize that they are acting as if markets are inefficient.

It would seem that the proper approach would be to select initial asset allocation proportions, purchase the associated fund shares, then make changes only as required when money is earned or needed for consumption. This makes considerable sense in the short run, but not in the longer run. Many things change in financial markets. Stocks pay dividends and bonds make interest payments. Some companies issue new stocks and others buy back some of their outstanding shares. New bonds are issued and existing ones mature or are called. New issuers of bonds and stocks come to market and some old ones disappear. Portfolios need to adapt to changes in the outstanding market capitalizations of asset classes.

The thrust of my Adaptive Asset Allocation Policies paper was that one should adapt to changing markets in a manner that would be *macro-consistent* – that is, if everyone adopted such a policy, markets would clear. In the case of our WBS portfolio, the appropriate policy is clear: one should periodically rebalance the proportions of the four asset classes to equal the outstanding market capitalizations at the time. Every investor could do this and markets would clear at the time. No bets against other investors would be made, either explicitly or implicitly.

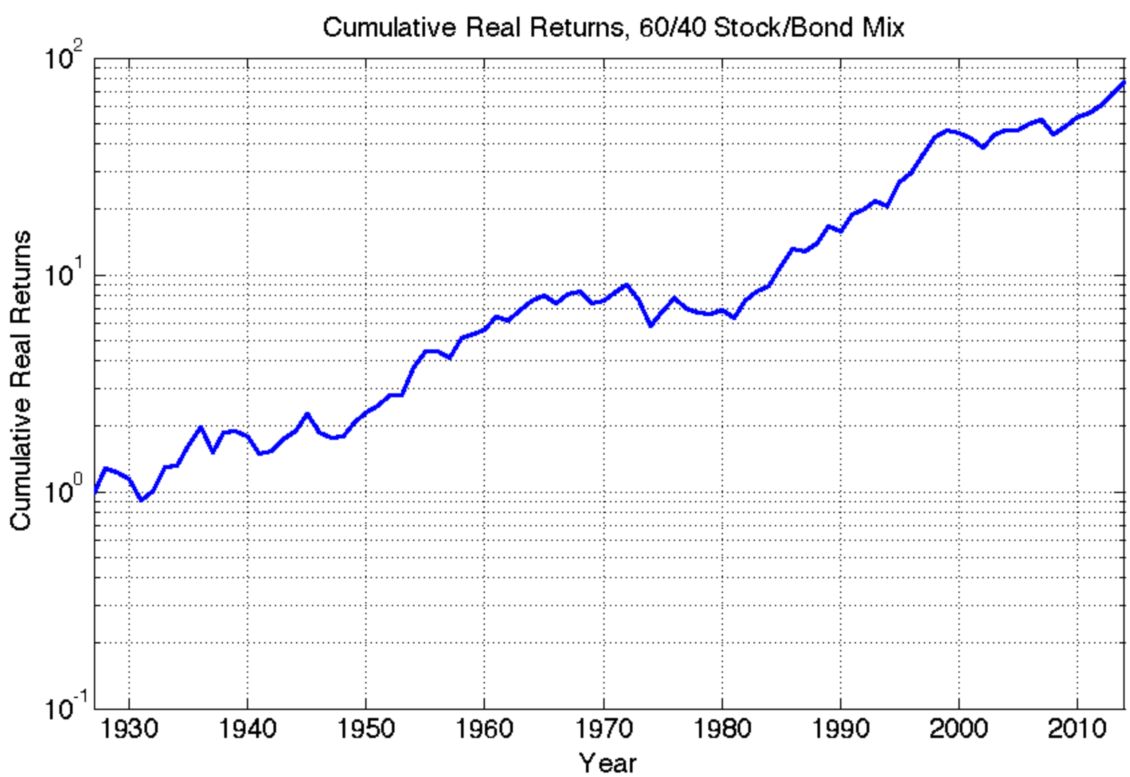
On a more practical level, it makes little sense to be obsessive about rebalancing. And of course it can only be done when the required market capitalization data become available – in our case, quarterly. It seems reasonable to make any needed changes with this frequency. To do so you can use the initialization program, setting the amount invested equal to the sum of the most recent values of the four component funds. Then, to obtain the cash amounts to be bought or sold for each fund, simply subtract the new amounts to be invested in the funds from the current values. Of course, the transactions will be completed using fund net asset values determined after markets close, so the proportions actually invested will likely differ slightly from those calculated. Nonetheless, the resulting holdings should provide a very close approximation of the proportionate values of the underlying indices.

Historic Bond and Stock Returns in the United States

We now have a proxy for the market portfolio of financial theory. But what should we assume about the probability distribution of its future returns? Unfortunately there is no easy answer. But it helps to begin by looking at the past.

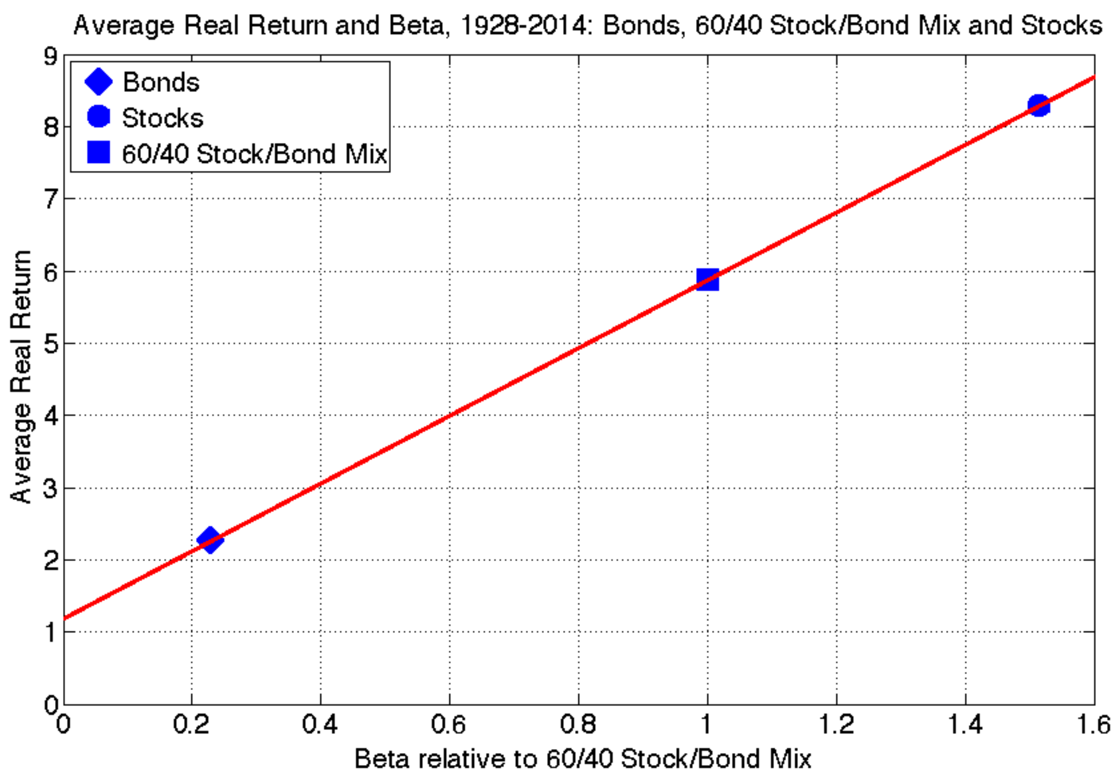
Annual data for the returns on Standard and Poor's 500-stock index and 10-year U.S. Treasury bonds from 1928 through the most recent year are available on a web site maintained by Aswath Damodaran at the Stern School of Business of New York University. Combining this information with data for the United States Consumer Price Index, taken from the website maintained by Robert Shiller of Yale University, provides information on the real returns from these two assets.

A very crude proxy for the market portfolio can be created by assuming that at the beginning of each year, the S&P500 comprised 60% of its value, with the remaining 40% invested in 10-year U.S. Treasury bonds. Dividing the value-relative for such a portfolio by the year-on-year ratio of the U.S. CPI gives the real return on a mix of U.S. stocks and bonds for each year from 1928 through 2014, expressed as a value-relative. The figure below shows the cumulative real value of $1 invested in this mix of stocks and bonds at the beginning of 1928, then rebalanced annually until the end of 2014.



Note that the vertical axis in the figure uses a logarithmic scale and the horizontal axis a regular scale, so that any given slope reflects the same percentage annual increase at any point in the graph. Clearly, the annual real returns varied from year to year. Real returns were negative in some years, close to zero in others, but positive more often than not. The worst year was 1973, when the portfolio's real value fell by 24%. The two market downturns in the current century were unpleasant, with the real value falling by roughly 17% from 1999 through 2002 and close to 14% in 2008. But the inclusion of bonds along with stocks greatly cushioned the blow, as stocks fell much more in each period. Overall, there was risk (the standard deviation of real return was 12.57%) but a reward for bearing it (the arithmetic average real return was 5.88%).

Given our assumption that 60/40 stock mix is a proxy for the market portfolio, we can create an *ex post* security market line using real returns our three assets; it is shown below.



Rather remarkably, this looks like a plausible choice for an *ex ante* security market line. The vertical intercept is 1.19% – close to our assumed riskless real return of 1.0%. And the slope is 4.68%, a plausible risk premium for a true market portfolio. (The fact that the three points lie on a straight line should not be a surprise, since both the beta value and average return of a combination of two assets will be proportionate to their relative holdings). Given the real return standard deviation of 12.57%, the Sharpe Ratio was 0.3723 (4.68/12.57).

Historic World Bond and Stock Returns

Of course, a portfolio of only U.S. securities may be a poor surrogate for a portfolio of world bonds and stocks. A broader view is provided by the monumental study of world historic returns in many countries reported in *Triumph of the Optimists, 101 Years of Global Investment returns* by Elroy Dimson, Paul Marsh and Mike Staunton (published by the Princeton University Press in 2002). They estimated that from 1900 through 2000, arithmetic average annual real returns were 7.2% for world equities and 1.7% percent for world bonds.This would suggest that a 60/40 Stock/Bond mix would have had an average real return of 5.0%, 4.0% greater than the average real return of 1.0% for U.S. Treasury bills.

The standard deviations of annual real returns were 17.0% for world stocks and 10.3% for world bonds. No statistics were given for their correlation. Although annual returns are not provided, there are results for real rates of return by decade. The correlation between such bond and stock real returns was 0.52. Using this as an estimate of the standard deviation of annual returns, the standard formula for the standard deviation of a portfolio of two assets gives an estimate of 12.83% per year.

Estimating Market Risk and Return

We now have two possible sets of empirical results that for the real returns on portfolios of bonds and stocks. The first three rows in the table below provide a summary.

	Risk Premium over Riskless Rate	Standard Deviation	Sharpe Ratio
U.S. Data	4.68	12.57	0.3723
World Data	4.00	12.71	0.3147
Average	4.34	12.64	0.3434
Estimates	4.25	12.50	0.3400

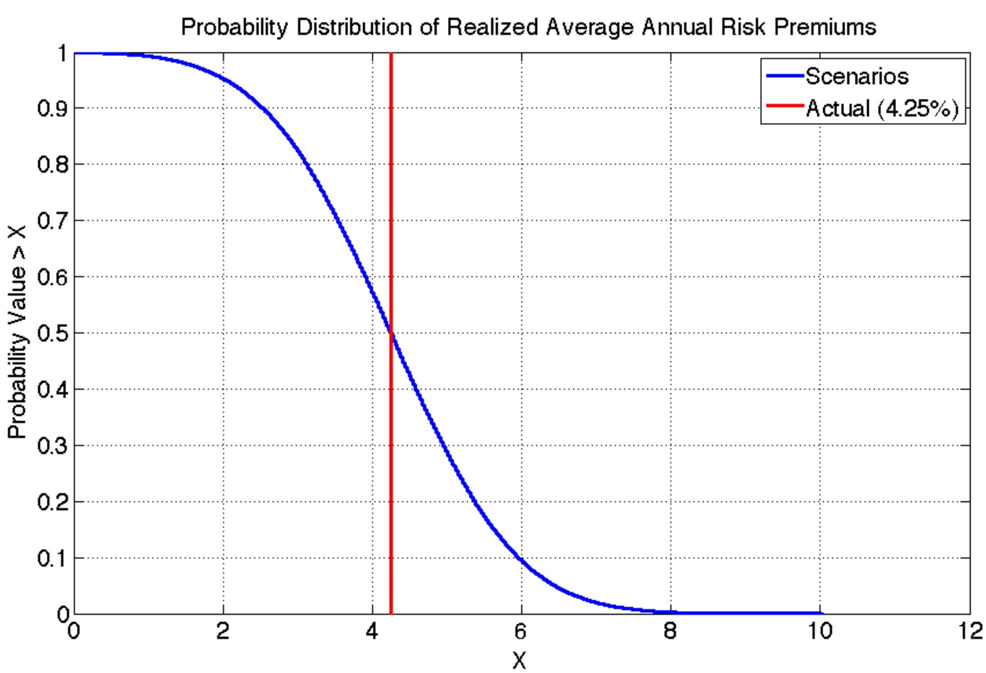
The final row, determined by rounding the averages to the nearest 0.25%, gives the values that we will use for the examples throughout this book.

Estimation Errors

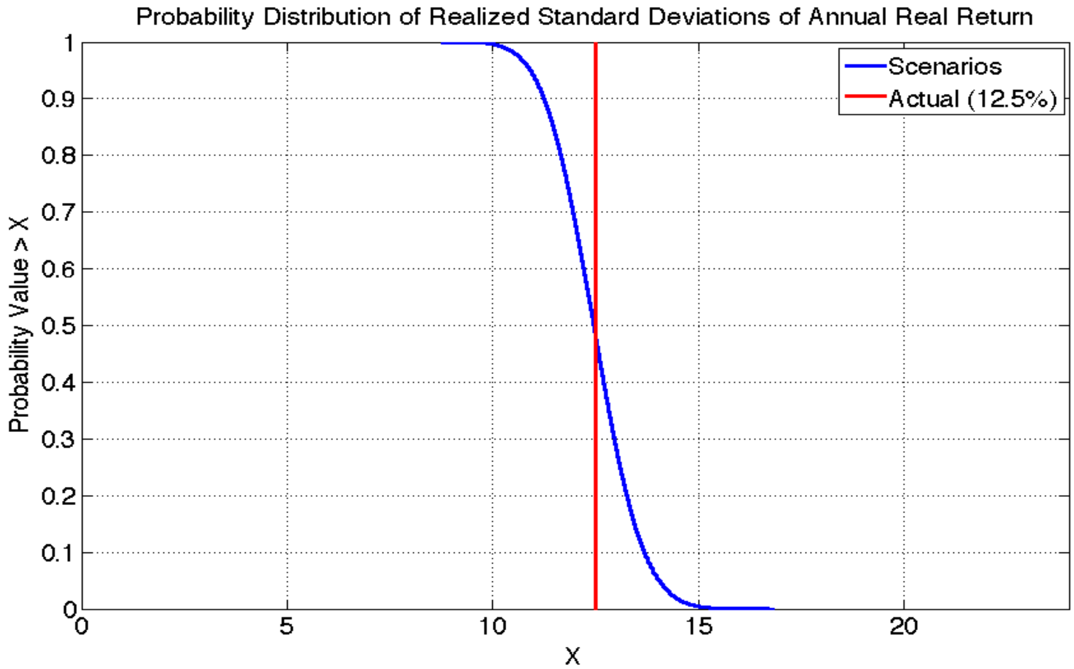
Of course the estimates we have shown are based on historic records for different areas over different time periods. Moreover, the portfolios utilized are far from our goal of one that includes all publicly traded liquid securities, or even the proxy for such a portfolio that we have constructed from four mutual funds. Moreover, the financial world is very different from that of the twentieth century or even the first part of the twenty-first century. A great deal of humility is required when making any forecasts of future investment returns, even probabilistic ones.

Some simulations can illustrate the dilemma. Assume that for each of 87 years, the real return on a portfolio over the riskless rate (risk premium) is drawn from an unchanging normal distribution with a mean of 4.25% and a standard deviation of 12.50%. At the end of the period, an economist will analyze the historic record, then estimate the future risk premium and standard deviation of the portfolio. How will he or she do? The answer depends, of course,on the realized returns over the 87 years, each of which has been drawn from the unchanging true distribution.

Consider first the task of estimating the expected future risk premium. The figure below shows the distribution of realized average returns across the 100,000 scenarios. While all results were generated from a distribution with a mean of 4.25%, the realized average risk premia varied from 1% to 8%. In 10% of the scenarios, the realized premium return was more than 6%, and in another 10% of the scenarios it was less than 3%; this despite the fact that results were generated by a process with an expected premium of 4.25%. But our economist sees only the historic results.



The situation is considerably better when it comes to estimating risk, as the next figure shows.



The likely estimation error for the standard deviation is smaller, but still substantial. This is a well-known property of probabilistic processes. Estimation errors are smaller for second moments, such as standard deviations, than for first moments, such as means. Moreover, by using smaller differencing intervals (for example, months rather than years), errors in estimating future standard deviations may be reduced, but this provides no help for estimating future expected returns.

Overall, the results of our exercise are depressing. Even if we were able to obtain a substantial history for the true market portfolio *and* if its returns had all been drawn from the same probability distribution every year *and* if future returns will be drawn from the same distribution, we could still make major errors when estimating parameters for the future distribution. And the reality is likely to be even worse. Political affairs, technology, communications, financial markets and financial economics have all changed radically over the last several decades. And they will undoubtedly change substantially in the future. When estimating future return distributions, humility is very much in order.

Geometric and Arithmetic Returns

A source of continuing confusion among users of return predictions (and sometimes providers) is the difference between arithmetic and geometric mean returns. This arises because cumulative value relatives are the *product* of periodic value relatives rather than their *sum*. To see this, assume that each year the market can go up 10% or down 10% with equal probability. Over a two-year period there are four possibilities:

(1.10*1.10) = 1.21

(1.10*0.90) = 0.99

(0.90*1.10) = 0.99

(0.90*0.90) = 0.81

The arithmetic mean (average) of the four ending values is 1.00. This is the expected two-year ending value. Taking the square root gives a one-year value relative of 1.00, with a return of 0.0% per year.

But consider the median ending value, which is 0.99. Taking the square root gives 0.995, for a one-year return of -0.50%. Note that the distribution is skewed to the left, so the mean is greater than the median.

A similar effect can be seen using the simulations we have already examined. We can summarize a long-term history by computing the constant return each year that would have produced the same ending value per dollar invested. Each of our simulated histories lasts 87 years so we take the 87'th root of the ending value divided by the beginning value to find the desired annual value relative. The average such value across the 100,000 simulations was 1.0348, equal to an annual return of 3.48% per year. This clearly differs from the simulated 4.25% annual return. Some would call this ex post constant return equivalent the *geometric mean return.* It can be estimated using a simple formula:

g = a – (sd^2) / 2

Here:

a -(sd^2) / 2 = .0425 - (0.125^2 / 2) = 0.0347 = 3.47%

Remarkably close to the mean of our simulated cases.

It is important to distinguish between these two concepts. In this case, the average return in a single year is 4.25%. And after 87 years, the arithmetic *mean* ending value was 36.9159, which could have been obtained with a constant annual return of 4.24% (close to the assumed mean annual return). But the *median* ending value is 19.7808, which could have been obtained with a constant annual return of 3.49%, close to our experienced and estimated geometric means.

People often mix up these two concepts. For example, one will sometimes hear that a portfolio with an assumed annual return of 7.5% will have a 50/50 chance of being worth (1.075^25) after 25 years. Wrong! That is the mean value. But since the ending value distribution will be highly skewed to the right, the median (50/50) value will be considerably less. A safe way to avoid such errors is to do what we have done – generate a large number of scenarios of future possible outcomes, then summarize the results appropriately.

Finally, when considering estimates of likely returns provided by financial analysts, it is important to understand the predictions being made. Some providers are careful to label their estimates as either arithmetic means or geometric means. Others may use terms like “long-run return estimate” for the latter. But in too many cases, there is confusion on the part of users and sometimes even producers.

Asset Allocation Recommendations

Before continuing, it is instructive to briefly consider some practical applications of future asset return distributions.

A number of financial consultants, investment banks, managers of large institutional funds and others, routinely produce estimates of expected returns, risks and correlations for multiple asset classes, then find “optimal” portfolios of such assets for different levels of risk tolerance. Procedures differ and are often considered proprietary, but a great deal of statistical analysis of past returns is usually employed, along with analysis of current conditions and sometimes estimates of future changes in markets and economies. Most make implicit or explicit assumptions that some asset classes are overpriced and others underpriced.

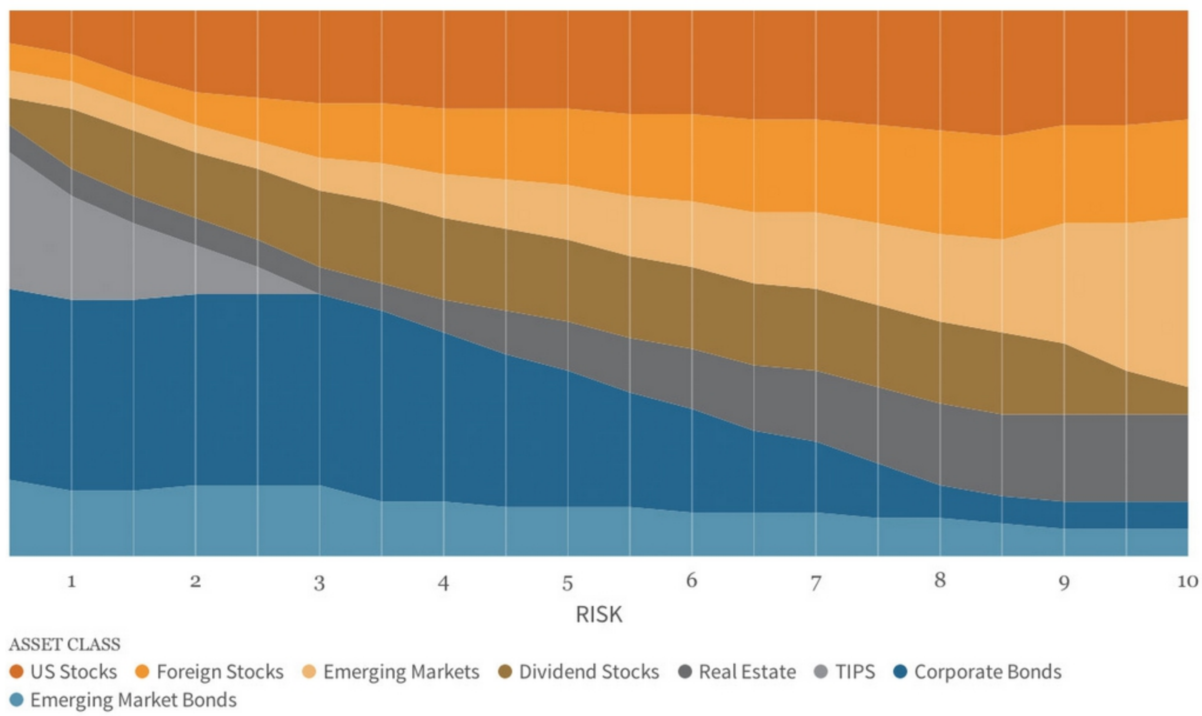
Some such firms estimate asset risks and correlations, assume a risk-free asset return and market expected return, and then use current asset market values to infer a set of asset expected returns consistent with an equilibrium that accords with the Capital Asset Pricing model. I first suggested such an approach (now termed *reverse optimization*) in “Imputing Expected Security Returns from Portfolio Composition, in the June 1974 J*ournal of Financial and Quantitative Analysis.* But many practitioners add another step to incorporate their own judgements about asset class mis-pricing, adjusting the equilibrium estimates to reflect such bets against the market. Some do this using a Bayesian statistical technique developed by Fischer Black and Robert Litterman, described in their paper on “Asset Allocation Combining Investor Views with Market Equilibrium”, in the September 1991 *Journal of Fixed Income.* The Black-Litterman procedure provides a set of estimates that combines prior estimates of expected returns with a firm's own views as well with an estimate of the importance to be accorded such such views, then produces a new set of expected returns that takes into account both forecasts and estimated risks and correlations.

However estimated, the asset risks, expected returns and correlations are then used as inputs to a portfolio optimization program, usually employing the mean/variance approach developed by Harry Markowitz, beginning with his 1952 *Journal of Finance* paper, Portfolio Selection”. The goal is to maximize portfolio expected return for each of a number of possible levels of risk, giving portfolios lying on the previously-described efficient frontier. Alternatively, one can choose to find the optimal efficient portfolio for a given risk tolerance (willingness to take on risk in order to increase expected return).

Despite the elegance of such an approach, it does have problems. As anyone who has experimented with portfolio optimization soon discovers, the recommended portfolios are incredibly sensitive to seemingly small changes in assumptions about relative asset expected returns. An optimization exercise with, say, 8 asset classes requires estimates of 28 different correlation coefficients, 8 standard deviations and 8 expected returns. As we have seen, under the best of circumstances it is very difficult indeed to estimate just two parameters for the probability distribution of the entire world market of bonds and stocks. The chance of doing so with any precision for 54 parameters for smaller segments of that market is diminishingly small.

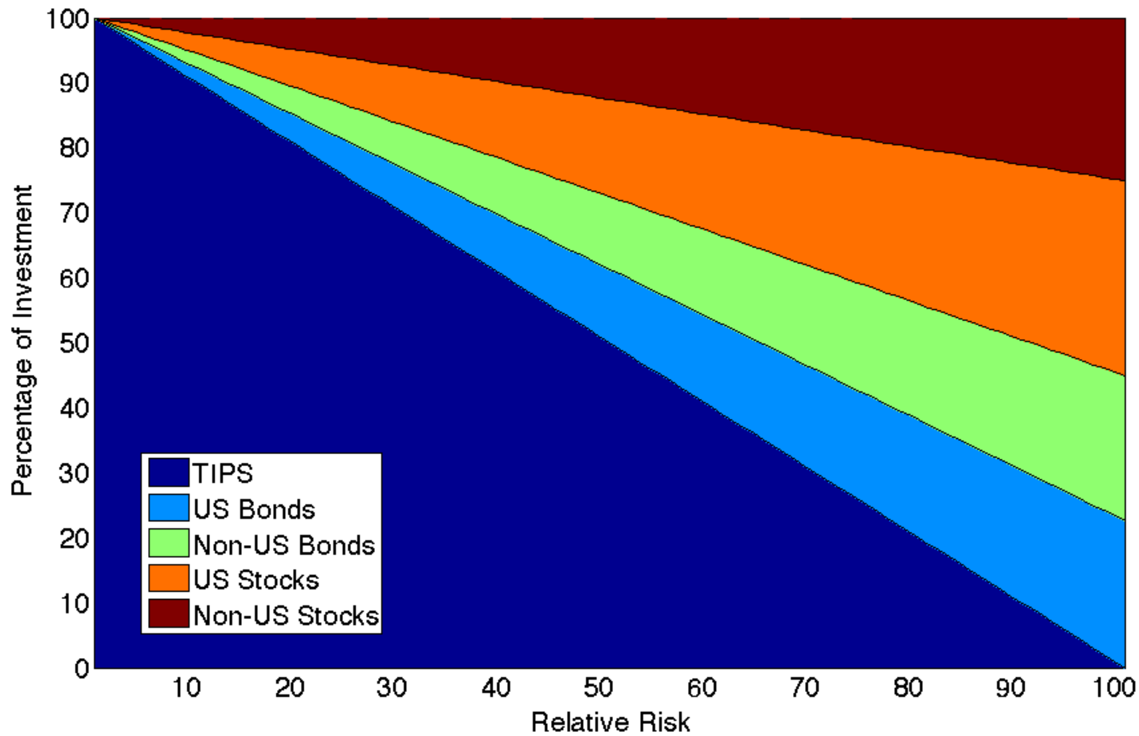
Worse yet without any constraints, optimizers are likely to choose portfolios with extremely large, small or even negative holdings. Indeed, it has been said that the quadratic programming methods used for portfolio optimization actually serve to maximize errors. To avoid ludicrous results, people who use such techniques either place upper and lower bounds on asset holdings or simply adjust the results of the optimizations to provide more palatable recommendations.

A very complete and readable description of a process that employs all these elements can be found in an *Investment Methodology White Paper* provided by Wealthfront, a “robo-advisor” offering online advice and portfolio management for individuals and institutions. The figure below shows the resulting recommended allocations among eight asset classes in July 2015 for tax-exempt investors willing to take on different amounts of risk (different recommendations are provided for investors paying taxes from investment returns and concerned about differential treatment of dividends and capital gains, etc.)



Interestingly, none of the asset mixes appears to reflect a combination of TIPS and a portfolio of world bonds and stocks in market proportions. And even the portfolio for the lowest risk category is relatively risky in real terms.

Contrast this with a diagram that might be produced by an investment advisor using TIPS and a mix of our four Vanguard funds and wishing to cover a broader range of risk levels. The figure below shows how it would have appeared in mid-2015, based on the market capitalizations shown earlier for June 30, 2015.



Far simpler, and based only on a rudimentary model of equilibrium. But not likely to generate 25 basis points in advisory fees.

In any event, these are the asset mixes we will consider for our investment strategies. More simply put, we will use only combinations of the market portfolio (shown here for a relative risk of 100) and TIPS (here, with a relative risk of 0). Moreover, since every mix is a combination of a risk-free security and a risky one, the standard deviation will be a linear function of the amount invested in the market mix. These mixes are fine for our purposes, and probably reasonable strategies for many retirees.

Market Return Distributions

Once the expected return of the market and its standard deviation of return have been estimated, it remains to specify the shape of the probability distribution. As with inflation, we will chose a lognormal distribution on the same grounds as those described in Chapter 5. Here is the argument .

First, the probability distribution of the sum of a series of random variables drawn from the same distribution will approach normality as the number of variables drawn increases. We know that the value relative of a return (for example, 1.02 for a 2% return) for a year will be the product of twelve monthly value relatives. Thus the logarithm of the value relative for a year will be the sum of the logarithms of twelve monthly value relatives. If the monthly value relatives are independently distributed, then the annual value relative will be approximately or exactly lognormally distributed. And, if *weekly* value relatives are independently distributed, the annual value relative distribution will be even closer to lognormal.

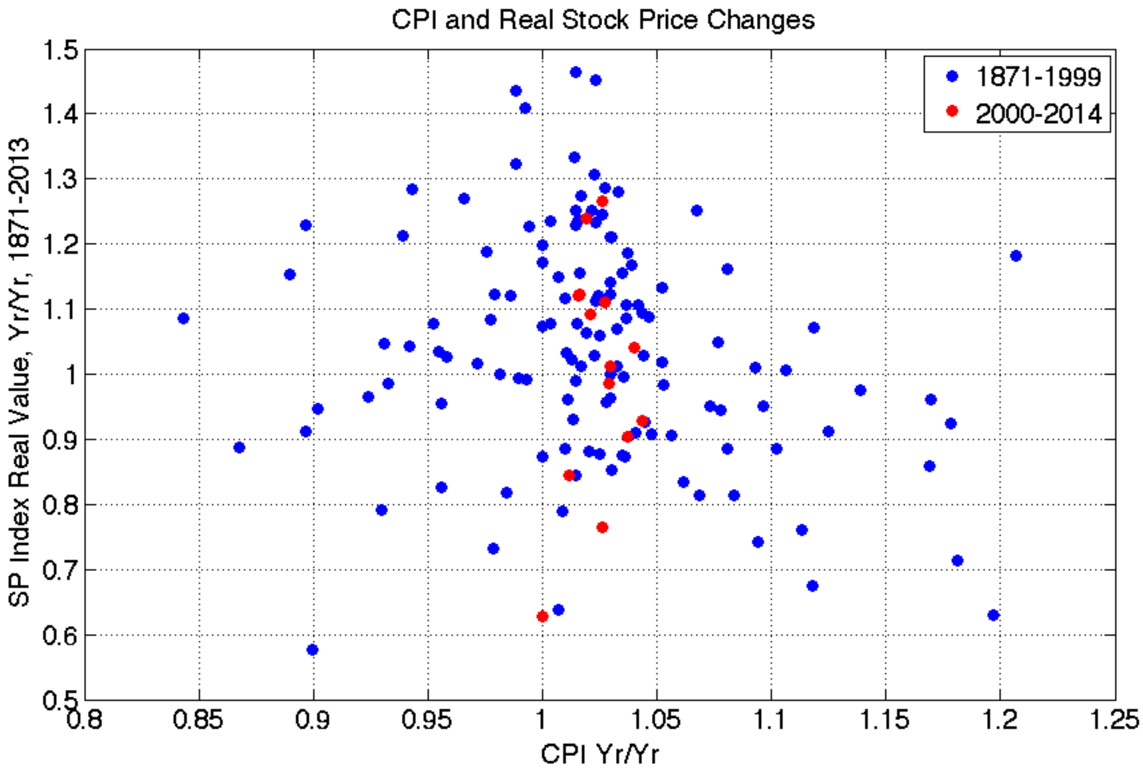
This is of course the same argument we made for assuming that inflation ratios are lognormally distributed. But the justification for assuming that monthly or weekly value relatives are independently distributed is far stronger here. The price of a traded security at any time reflects a sort of consensus opinion about the future prospects of its issuer. Any significant change from one time to the next will typically be due to new information (news) not incorporated in the previous price. For example, the probability distribution of the market return in January will reflect the effects of possible news relevant for the value of the market during the month of January. And the probability distribution of the market return in February will reflect the effects of possible news during the month of February. To belabor the obvious: news is *new*. The prices of securities on January 1st reflect expectations at that time for the near and distant future, based on information available at the time . The prices on February 1st reflect expectations at that time, based on information available at the time. Whatever the shapes of the probability distributions for returns in January and February may be, they are likely to be relatively independent. And the central limit theorem leads to the conclusion that their products are likely to be lognormally distributed.

Such is the argument for assuming that the market value relative for a year should be drawn from a lognormal distribution. But the same argument about the impact of new news can be used to argue that the distribution should be the same for every future year. We thus choose to model annual real returns on the market portfolio as *independent and identically lognormally distributed (i.i.d),* with the parameters chosen earlier.

This may seem overly simplistic, and perhaps it is. A number of financial analysts have tortured historic data sufficiently to derive justifications for far more complex distribution assumptions. Some advocate assuming that return distributions have "fat left tails" with substantial probabilities of disastrous outcomes. Others believe they can predict higher than normal ranges of return at some times and lower at other times, depending on recent history. On closer examination, many of these assumptions implicitly or explicitly assume that markets do not take existing information about firms and economies fully into account at all times. But the profit motive is strong among investors; moreover, capital markets are highly competitive, so the assumption that returns are independently distributed does not seem demonstrably wrong. Moreover, the simulations in our earlier section on estimation errors suggest that in this respect as well, the past may be a poor prologue for the future. For better or worse, we choose i.i.d. lognormally distributed annual market returns.

Market Returns and Inflation

One last question needs to be addressed before we turn to programs. Are future market real returns likely to be correlated with levels of inflation? The following figure, provided by Robert Shiller, compares annual values of the year-over-year ratios of the CPI with those for the Standard and Poor's stock index from 1871 through 2014. There is a negative relationship for the years prior to 2000 but it is only barely statistically significant, with a t-statistic of -2.21. For the first fifteen years of the twenty-first century the relationship is slightly positive, but insignificantly so, with a t-statistic of +0.08.



At the very least, there is little evidence to support an assumption of a correlation between changes in the CPI and real returns. Accordingly, we will generate market real returns and levels of inflation separately.

Generating Market Returns

We turn finally to the program statements required to create the market portfolio components of our market structure. We add the following statements to the **market_create** function introduced in Chapter 5:

```
% market portfolio returns
   market.exRm    = 1.0425;  % market portfolio expected return over risk-free rate
   market.sdRm    =  0.125;   % market portfolio standard deviation of return
```

These provide the default values for the expected annual excess return on the market portfolio and its standard deviation.

Next we add to our **market_process( )** function, statements similar to those employed for inflation in order to produce matrices of market returns and cumulative returns at the beginning of each year:

```
% compute market returns matrix
  u = market.exRm + ( market.rf – 1 ); % total expected return
  v = market.sdRm^2;      % variance
  b  = sqrt( log(( v / (u^2) ) + 1) );
  a  = 0.5 *log( (u^2) / exp(b^2) );
  market.rmsM = exp(a + b*randn( nrows,ncols ));
 % compute market cumulative returns matrix
  m = cumprod( market.rmsM , 2 );
  market.cumRmsM = [ ones( nrows , 1 )  m( : , 1:ncols-1 ) ];
```

With these additions, the **market_process( )** function will produce matrices for annual and cumulative values of (1) the cost of living, (2) risk-free returns and (3) market returns. And it will do so quickly (on the author's MacBook, in less than half a second for 100,000 scenarios).